Risk Aversion, Wealth Inequality and Portfolio Choice

Applications of data science to portfolio optimisation

Cherry Muijsson and Steve Satchell

Faculty of Economics
Two questions

• Are wealthier individuals less risk averse?

• Are less risk averse individuals prone to riskier investment decisions and do they choose riskier portfolios?
Why do these questions matter?

• In *Capital in the Twenty-First Century* (2014), Thomas Piketty describes the growing concentration of wealth among the very few and proposes a global coordinated wealth tax to counter the heavily skewed distribution.

• Population ageing could change the distribution of wealth towards older generations, with more conservative investment preferences.

• Do changes in the distribution of wealth affect the distribution of risk aversion, and hence aggregate portfolio choice?
Wealthier individuals may show increased participation on the stock market, and may hire professional investors to act on their behalf.

The equity premium puzzle (Mehra and Prescott 1985) refers to excess returns on stocks over the past century being much higher than returns on government bonds.

Paiella (2004) finds that representative stockholding agents are less risk averse than representative agents, and shows that the equity risk premium is more realistic when stockholding investors are considered instead.
Risk aversion and wealth: introduction (2)

• The equity return in excess of the market is dependent on the definition of the market portfolio.

• What is the market portfolio when the distribution of risk aversion is conditional on the distribution of individual wealth?

• Brennan (1971) recognises the effect of heterogeneous preferences on the excess returns of a security, and allows for investors facing different borrowing and lending rates. He finds that investors would hold vastly different portfolios.

• We derive the optimal market portfolio in a universe of heterogeneous investors with a distribution of risk aversion conditional on the distribution of wealth, and show the impact on the equity risk premium.
Roadmap

• Brief discussion of main concepts
• Theoretical framework
• Empirical results
• Concluding remarks and recommendations
• Discussion
Decreasing absolute risk aversion

- Risk aversion determines the optimal portfolio for an agent (Sharpe 1978)
- The absolute risk aversion ($\lambda$) of an agent is determined by the shape of their utility function $U(W)$ and captured by the Arrow-Pratt measure.
ARA is constant in wealth (CARA) in most models, but models could allow for decreasing or increasing absolute risk aversion (DARA and IARA respectively).


The joint distribution of risk aversion and wealth can be written as follows:

$$\int_0^1 \int_0^\infty pdf(\omega, \lambda)d\lambda d\omega$$
Theoretical framework – risk aversion

source: Muijssen and Satchell (2016)
Theoretical framework – wealth proportions

source: Muijsson and Satchell (2016)
The Markowitz portfolio

• Main method to derive the optimal portfolio consisting of a set of securities by selecting an optimal allocation based on a risk-return tradeoff

• The optimal portfolio weights are a direct function of the covariances of the set of securities, their market exposure, and investor risk aversion

• The vector of optimal portfolio weights is

\[ \phi = \frac{\Sigma^{-1}(\mu - r_f)}{\beta - r_f\gamma} = \frac{\Sigma^{-1}(\mu - r_f)}{\lambda} \]

• Here, \( \mu \) is a vector of mean asset returns with covariance matrix \( \Sigma \), and it can be shown that the denominator is equal to the coefficient of ARA, \( \lambda \).
The Markowitz portfolio

source: Muijsson and Satchell (2016)
The equity risk premium

• The risk premium is related to the mean portfolio return ($\mu_m$) and risk aversion:

\[
\text{risk premium} = \mu_m - r_f
\]

• The cost of capital is similarly affected by the mean portfolio return:

\[
\text{cost of capital} = r_f + \beta_i(\mu_m - r_f)
\]

• There is an incentive to increase the cost of capital to justify increasing dividends:

\[
r = \frac{D \, (= \, Q_d \times P_d)}{r_f + \beta(\mu_m - r_f)}
\]
Theoretical framework – risk aversion

• Risk aversion drives portfolio choice

• Distinguish two types of portfolios: borrowing and lending

• Borrowing and lending investors have different interest rates, $r_b$ and $r_l$

• Borrowing investors go short on cash and invest in equity, while lending investors go long in both cash and equity.

• Mixed portfolios are possible
Theoretical framework – multiple optimal portfolios

source: Muijspoon and Satchell (2016)
Theoretical framework – investor types

• Distinct borrowing and lending portfolios, where \( r_b > r_l \):

\[
\phi_b = \frac{\Sigma^{-1}(\mu - r_{bi})}{\beta - r_b\gamma} \quad \text{and} \quad \phi_l = \frac{\Sigma^{-1}(\mu - r_{li})}{\beta - r_l\gamma}
\]

• Investors are distributed in three categories (lending, mixed, borrowing) based on their absolute risk aversion

• Theorem 1: boundary conditions

\[
\begin{align*}
\lambda &\geq \beta - r_l\gamma & \text{lending investor} \\
\beta - r_b\gamma &< \lambda < \beta - r_l\gamma & \text{mixed investor} \\
\lambda &\leq \beta - r_b\gamma & \text{borrowing investor}
\end{align*}
\]

• The market portfolio should reflect equity wealth shares of investors and their portfolios based on the risk aversion shares
Theoretical framework – weighted market portfolio

• Theorem 2: the aggregate cash position is \((c_1 + c_2)\), and the market portfolio is:

\[ M = (a_1 + a_2)\phi_l + (b_1 + b_2)\phi_b \]

• The shares are determined from a distribution:

\[
a_1 = \int_{\beta - r_l \gamma}^{\infty} \int_0^1 \frac{\beta - r_l \gamma}{\lambda} pdf(\omega, \lambda)d\omega d\lambda
\]

\[
a_2 = \int_{\beta - r_b \gamma}^{\beta - r_l \gamma} \int_0^1 \frac{\beta - r_l \gamma}{\gamma(r_b - r_l)} \left(1 - \frac{\beta - r_b \gamma}{\lambda}\right) pdf(\omega, \lambda)d\omega d\lambda
\]

\[
b_1 = \int_{\beta - r_b \gamma}^{\beta - r_l \gamma} \int_0^1 1 - \left(\frac{\beta - r_l \gamma}{\gamma(r_b - r_l)} \left(1 - \frac{\beta - r_b \gamma}{\lambda}\right)\right) pdf(\omega, \lambda)d\omega d\lambda
\]

\[
b_2 = \int_0^{\beta - r_b \gamma} \int_0^1 \frac{\beta - r_b \gamma}{\lambda} pdf(\omega, \lambda)d\omega d\lambda
\]
Theoretical framework – distributions

• The cash position is \((c_1 + c_2)\), the market portfolio is \(M = (a_1 + a_2)\phi_l + (b_1 + b_2)\phi_b\)

• Wealth is a sum of equity and cash holdings for each type of investor:

  Lending investor wealth (long in cash and equity): \(e_1 = a_1 + c_1\)
  Mixed investor wealth (long in both portfolios): \(e_2 = a_2 + b_1\)
  Borrowing investor wealth (long in equity, short in cash): \(e_3 = b_2 + c_2\)

• The choice of distribution influences the investor type and wealth shares:

  Conditional beta distribution:
  
  \[
pdf(\omega|\lambda) = \frac{\Gamma(\delta(\lambda)+\nu(\lambda))}{\Gamma(\delta(\lambda))\Gamma(\nu(\lambda))} \omega^{\delta(\lambda)-1}(1 - \omega^{\nu(\lambda)-1})
  \]

  Bivariate lognormal distribution:
  
  \[
pdf(\omega, \lambda) = \frac{1}{2\pi\sigma_\omega\sigma_\lambda\sqrt{1-\rho^2}} \exp\left(-\frac{2}{2(1-\rho^2)}\right)
  \]

• Correlation determines the result in both distributions
Theoretical framework – distributions

Beta Density Function

Gamma Density Function

source: Wikipedia graphs
Data Description

- Annualized daily returns from January 1995 until December 2004 for five major indices: AEX, DAX, FTSE, NKY, S&P
- One year LIBOR as risk free rate

source: Muijsson and Satchell (2016)
Results 1: optimal portfolios

- The Markowitz and lending portfolio are equivalent in this case, as the lending rate is LIBOR and the gap between the borrowing and lending rate is 2.5%.
- As expected, there are no short positions in the lending portfolio.
- The borrowing portfolio goes short on the FTSE during this period.

Table 1: Portfolios Corresponding to Indices and Investor Proportions

<table>
<thead>
<tr>
<th></th>
<th>Markowitz</th>
<th>Lending</th>
<th>Borrowing</th>
<th>Market</th>
</tr>
</thead>
<tbody>
<tr>
<td>AEX</td>
<td>0.0163</td>
<td>0.0163</td>
<td>0.0786</td>
<td>0.0426</td>
</tr>
<tr>
<td>DAX</td>
<td>0.2220</td>
<td>0.2220</td>
<td>0.3486</td>
<td>0.2756</td>
</tr>
<tr>
<td>UKX</td>
<td>0.0510</td>
<td>0.0510</td>
<td>-0.2145</td>
<td>-0.0613</td>
</tr>
<tr>
<td>NKY</td>
<td>0.3464</td>
<td>0.3464</td>
<td>0.4379</td>
<td>0.3851</td>
</tr>
<tr>
<td>SPX</td>
<td>0.3642</td>
<td>0.3642</td>
<td>0.3493</td>
<td>0.3579</td>
</tr>
</tbody>
</table>

source: Muijsson and Satchell (2016)
Results 2: portfolio proportions when $\rho < 0$

- The conditional beta distribution is calibrated to have a correlation of -0.40
- The share of lending investors is 57.8%, they hold 66.8% of wealth
- Mixed investors are 11.5% of the total, and hold 9.7% of total wealth

<table>
<thead>
<tr>
<th>Investor Type</th>
<th>Proportion in Equity</th>
<th>Proportion in Cash</th>
<th>Wealth Proportion</th>
<th>Investor Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lending</td>
<td>0.4526</td>
<td>0.2154</td>
<td>0.6680</td>
<td>1.1077</td>
</tr>
<tr>
<td>Mixed</td>
<td>0.0970</td>
<td>0</td>
<td>0.0970</td>
<td>0.9151</td>
</tr>
<tr>
<td>Borrow</td>
<td>0.3850</td>
<td>-0.1499</td>
<td>0.2350</td>
<td>0.8077</td>
</tr>
</tbody>
</table>

Ratios are defined as the wealth proportion per investor group over the total share of these investors in the population. Parameters for gamma distribution for risk aversion are $\theta = 6$ and $k = 5$. Parameters of conditional beta are $\delta_1 = 0.5$, $\nu_2 = 1$, $\rho = -0.40$.

source: Muijssoen and Satchell (2016)
Results 2: portfolio proportions when \( \rho > 0 \)

- The conditional beta distribution is calibrated to have a correlation of 0.76
- The share of lending (borrowing) investors is 58.4%, they hold 63.6% of wealth
- Mixed investors are 9.4% of the total, and hold 10.2% of total wealth
- Mixed investors are better off when correlation is positive!

<table>
<thead>
<tr>
<th></th>
<th>Proportion Investors</th>
<th>Proportion in Equity</th>
<th>Proportion in Cash</th>
<th>Wealth Proportion</th>
<th>Investor Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lending</td>
<td>0.5840</td>
<td>0.4210</td>
<td>0.2154</td>
<td>0.6364</td>
<td>1.0897</td>
</tr>
<tr>
<td>Mixed</td>
<td>0.0940</td>
<td>0.1020</td>
<td>0</td>
<td>0.1020</td>
<td>1.0851</td>
</tr>
<tr>
<td>Borrow</td>
<td>0.3220</td>
<td>0.4074</td>
<td>-0.1458</td>
<td>0.2616</td>
<td>0.8124</td>
</tr>
</tbody>
</table>

Parameters for gamma distribution for risk aversion are \( \theta = 6 \) and \( k = 5 \). Parameters of conditional beta are \( \delta_1 = 2, \quad \nu_2 = 1, \quad \rho = 0.76 \).

source: Muijssen and Satchell (2016)
Results 3: professional investor universe

Table 3: Investor Shares for Professional Investor Universe with Positive Correlation

<table>
<thead>
<tr>
<th></th>
<th>Proportion Investors</th>
<th>Proportion in Equity</th>
<th>Proportion in Cash</th>
<th>Wealth Proportion</th>
<th>Investor Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lending</td>
<td>0.4370</td>
<td>0.0763</td>
<td>0.1909</td>
<td>0.2672</td>
<td>0.6114</td>
</tr>
<tr>
<td>Mixed</td>
<td>0.1050</td>
<td>0.1020</td>
<td>0</td>
<td>0.1020</td>
<td>0.9714</td>
</tr>
<tr>
<td>Borrow</td>
<td>0.4580</td>
<td>0.7093</td>
<td>-0.0785</td>
<td>0.6308</td>
<td>1.3773</td>
</tr>
</tbody>
</table>

Table 3b: Investor Shares for Professional Investor Universe with Negative Correlation

<table>
<thead>
<tr>
<th></th>
<th>Proportion Investors</th>
<th>Proportion in Equity</th>
<th>Proportion in Cash</th>
<th>Wealth Proportion</th>
<th>Investor Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lending</td>
<td>0.4290</td>
<td>0.1103</td>
<td>0.1955</td>
<td>0.3058</td>
<td>0.7128</td>
</tr>
<tr>
<td>Mixed</td>
<td>0.1120</td>
<td>0.1180</td>
<td>0</td>
<td>0.1180</td>
<td>1.0536</td>
</tr>
<tr>
<td>Borrow</td>
<td>0.4590</td>
<td>0.6633</td>
<td>-0.0871</td>
<td>0.5762</td>
<td>1.2554</td>
</tr>
</tbody>
</table>

Source: Muijssen and Satchell (2016)

- Mixed investors are better off when correlation is negative!
- Effect depends on whether lending investors form the largest share in market
Results 4: bivariate lognormal correlation cases

Table 4: Investor Shares for Bivariate Lognormal with Negative Correlation $\rho = -0.40$.

<table>
<thead>
<tr>
<th></th>
<th>Proportion in Equity</th>
<th>Proportion in Cash</th>
<th>Wealth Proportion</th>
<th>Investor Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lending</td>
<td>0.7760</td>
<td>0.0231</td>
<td>0.8095</td>
<td>1.0432</td>
</tr>
<tr>
<td>Mixed</td>
<td>0.1288</td>
<td>0</td>
<td>0.1124</td>
<td>0.8727</td>
</tr>
<tr>
<td>Borrow</td>
<td>0.0952</td>
<td>-0.0660</td>
<td>0.0781</td>
<td>0.8205</td>
</tr>
</tbody>
</table>

Parameters for bivariate normal distribution are $\mu_w = 340,000$, $\sigma_w^2 = 640,000$, $\mu_\lambda = 30$, $\sigma_\lambda^2 = 50$, $\rho = -0.40$.

Table 4b: Investor Shares for Bivariate Lognormal with Positive Correlation $\rho = 0.76$.

<table>
<thead>
<tr>
<th></th>
<th>Proportion in Equity</th>
<th>Proportion in Cash</th>
<th>Wealth Proportion</th>
<th>Investor Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lending</td>
<td>0.7735</td>
<td>0.0251</td>
<td>0.8104</td>
<td>1.0477</td>
</tr>
<tr>
<td>Mixed</td>
<td>0.1328</td>
<td>0</td>
<td>0.1132</td>
<td>0.8524</td>
</tr>
<tr>
<td>Borrow</td>
<td>0.0937</td>
<td>0.0022</td>
<td>0.0764</td>
<td>0.8155</td>
</tr>
</tbody>
</table>

Parameters for bivariate normal distribution are $\mu_w = 340,000$, $\sigma_w^2 = 640,000$, $\mu_\lambda = 30$, $\sigma_\lambda^2 = 50$, $\rho = 0.76$.

source: Muijsson and Satchell (2016)
Results 5: market portfolios for all cases

- The market portfolios are highly dependent on the correlation, choice of distribution and the share of investor types in the market
- Particularly the conditional beta specification is sensitive to correlation
- The market portfolio with the shortest position is for the negative conditional beta

Table 5: Market Portfolios for all Distribution, Correlation and Universe Scenarios

<table>
<thead>
<tr>
<th></th>
<th>Marko</th>
<th>Beta(+)</th>
<th>Beta (-)</th>
<th>Bivar(+)</th>
<th>Bivar (-)</th>
<th>Prof (+)</th>
<th>Prof (-)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AEX</td>
<td>0.0163</td>
<td>0.0479</td>
<td>0.0644</td>
<td>0.0509</td>
<td>0.0618</td>
<td>0.0598</td>
<td>0.0602</td>
</tr>
<tr>
<td>DAX</td>
<td>0.2220</td>
<td>0.2862</td>
<td>0.3198</td>
<td>0.2924</td>
<td>0.3146</td>
<td>0.3103</td>
<td>0.3111</td>
</tr>
<tr>
<td>UKX</td>
<td>0.0510</td>
<td>-0.0837</td>
<td>-0.1542</td>
<td>-0.0966</td>
<td>-0.1431</td>
<td>-0.1342</td>
<td>-0.1360</td>
</tr>
<tr>
<td>NKY</td>
<td>0.3464</td>
<td>0.3928</td>
<td>0.4171</td>
<td>0.3973</td>
<td>0.4133</td>
<td>0.4102</td>
<td>0.4108</td>
</tr>
<tr>
<td>SPX</td>
<td>0.3642</td>
<td>0.3566</td>
<td>0.3527</td>
<td>0.3559</td>
<td>0.3533</td>
<td>0.3538</td>
<td>0.3537</td>
</tr>
</tbody>
</table>

Beta refers to results for conditional beta distribution, Bivar to results for bivariate lognormal distribution, and Prof to results for conditional beta calibrated for professional investor universe. (+) refers to correlation $\rho = 0.76$ and (-) to $\rho = -0.40$.

source: Muijssen and Satchell (2016)
Results 6: equity risk premium for all cases

• The mean portfolio returns for the heterogeneous cases are significantly higher than the mean returns on the Markowitz portfolio.

• The equity risk premium for LIBOR (lending rate) ranges from 7.5% and 9.2%, while the borrowing rate ERP ranges from 5.1% and 6.7%.

• The mixed rate is determined based on investor shares, and leads to a more robust result.

Table 6: Equity Risk Premia for all Interest Rate Cases and Market Portfolio Scenarios

<table>
<thead>
<tr>
<th>Case</th>
<th>( \mu_p )</th>
<th>( r_l )</th>
<th>( r_b )</th>
<th>( r_{mixed} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Markowitz</td>
<td>0.1312</td>
<td>0.0756</td>
<td>0.0506</td>
<td>0.0623</td>
</tr>
<tr>
<td>Beta (+)</td>
<td>0.1419</td>
<td>0.0863</td>
<td>0.0613</td>
<td>0.0730</td>
</tr>
<tr>
<td>Beta (-)</td>
<td>0.1476</td>
<td>0.092</td>
<td>0.067</td>
<td>0.0787</td>
</tr>
<tr>
<td>Bivar (+)</td>
<td>0.1430</td>
<td>0.0874</td>
<td>0.0624</td>
<td>0.0741</td>
</tr>
<tr>
<td>Bivar (-)</td>
<td>0.1467</td>
<td>0.0911</td>
<td>0.0661</td>
<td>0.0778</td>
</tr>
<tr>
<td>Prof (+)</td>
<td>0.1460</td>
<td>0.0904</td>
<td>0.0654</td>
<td>0.0771</td>
</tr>
<tr>
<td>Prof (-)</td>
<td>0.1461</td>
<td>0.0905</td>
<td>0.0655</td>
<td>0.0772</td>
</tr>
</tbody>
</table>

source: Muijsson and Satchell (2016)
Concluding remarks and discussion

- The distribution of risk appetite determines the market portfolio
- Investor relative wealth depends on the correlation between risk aversion and wealth
- When lending investors dominate the market, investors that invest in both the borrowing and lending portfolio are better off in terms of their wealth share when wealth is decreasing in absolute risk aversion
- The equity risk premium is far more robust and consistent with empirical evidence when it is calculated with an interest rate that is weighted by investor type proportions